

## ON INFINITE REGIONS\*

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The object of the present paper is to lay down a general definition of infinite regions which will include the cases of projective geometry, the geometry of inversion, the geometry of the space of analysis, and other geometries which, like these, may be based on a set of elements consisting of the points of ordinary space of  $n$  complex dimensions, extended by a complex  $(n - 1)$ -dimensional set of ideal elements—the so-called “points at infinity.” This latter manifold is algebraic in character, being in the familiar cases the line or plane at infinity, the null circle or sphere at infinity, or the  $n$  hyperplanes at infinity.

We are not concerned directly with geometries based on ordinary real space—real projective geometry, the real geometry of inversion, etc. In these geometries the infinite region is sometimes a manifold of only one less dimension than the space it closes, and the extended space is then not necessarily linearly simply connected. Here, however, the space considered is of  $2n$  real dimensions, and it is closed by a manifold of two less dimensions, the resulting extended space being always linearly simply connected.

A prototype for one class of cases included here is given by the geometry of inversion, to which reference has already been made. In that geometry a one-to-one relation is established between the elements of the group of circular transformations, whose transformations operate on the points of complex  $n$ -dimensional space, and the group of collineations in space of one higher dimension, which carry a certain quadric surface (or hypersurface, when  $n > 2$ ) over into itself. Furthermore, a one-to-one relation between the points of the  $n$ -dimensional space and the points of the quadric is established. Now the quadric is taken in projective space, and it is closed by the ideal points in which it meets the plane at infinity. But ordinary  $n$ -dimensional space is not closed, and it is not until it has been closed by a suitable set of ideal elements—the so-called *infinite region*—that such a correspondence is possible. Aside, however, altogether from a correspondence of this kind, it is necessary to introduce the same set of ideal elements if the transformations of the first named group are to carry each point of the set on which they operate into a point of that set.

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The immediate object of the present paper is, however, not geometric, but analytic. Weierstrass stated the theorem that a function of  $n$  complex variables which is meromorphic at every point of space is rational; and he understood by *space* the extended space of analysis—the  $n$  spheres of the  $n$  complex variables. It was for this case that Hurwitz gave the proof. But the theorem is true for other spaces,\* and it is the purpose of this paper to show that it holds for all spaces closed by an infinite region according to the definitions which follow.

With the attainment of this end a gap in the theory of functions of several complex variables has been filled and the purpose of this investigation accomplished. But the formulation of the definition here considered prepares the way for the statement of an algebraic-geometrical problem of great generality, namely: To determine all extensions of the ordinary space of  $n$  complex variables by an infinite region, of  $n - 1$  complex dimensions, such that the extended space admits a transformation into itself which is one-to-one and regular at every point; and for a given space to determine all such transformations.† The present paper contributes to this problem the result that all such transformations are birational.

One final remark. The introduction of an infinite region is a *convenience*, not a *necessity*. It is the transformations—in practice, a group of transformations—and invariants with respect to the same that are the essential thing, and the formulation of the theorems can always be so modified as to retain the substance and still preserve accuracy. Thus in the case of Weierstrass's theorem, § 2 below could be omitted altogether. In § 3 we should then demand that the function  $\Omega(z_1, \dots, z_n)$  go over by each of the transformations corresponding to a set of functions ( $F$ ) belonging to a given class  $\mathfrak{R}$  into a new function  $\Omega'(z'_1, \dots, z'_n)$  meromorphic except perhaps for removable singularities in the neighborhood of a point  $A'$ . Such a function is then shown to be rational by the same analysis as that used in § 3.

### 1. ON A CERTAIN CLASS OF FUNCTIONS

*Condition A.* Let each of the functions  $F_i(z'_1, \dots, z'_n)$ ,  $i = 1, \dots, n$ , be meromorphic in the finite point  $A' = (a'_1, \dots, a'_n) = (a')$ , and let at least one of them have a singular point there. Make the transformation

$$(1) \quad z_i = F_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n).$$

\* A first generalization for the case of projective space was given by the author in these *Transactions*, vol. 13 (1912), p. 159, and further extensions for the intermediate spaces (cf. *Madison Colloquium*, p. 141) have been obtained by Dr. Jackson, in a forthcoming number of the *Journal für Mathematik*.

† For projective space and the space of analysis it is readily shown that the totality of such transformations is given by the linear transformations.

If these functions are so constituted that they carry that part  $\sigma'$  of a certain neighborhood  $s'$  of  $A'$ , in which they are all analytic, in a one-to-one manner over into a region  $\sigma$ , then they shall be said to satisfy *Condition A*.

More precisely, they satisfy *Condition A at the point  $A'$* . It is evident that they also satisfy *Condition A* at every interior point of  $s'$  which does not lie in  $\sigma'$ ; i. e., in which at least one of the functions has a singularity.

We shall frequently denote a set of  $n$  such functions  $F_i$  by  $(F)$ .

A transformation

$$(2) \quad w_i = \omega_i(z_1, \dots, z_n) \quad (i = 1, \dots, n)$$

shall be said to be *regular* at a point  $(z) = (a)$  if each of the functions  $\omega_i$  is analytic at  $(a)$  and the jacobian

$$\frac{\partial(\omega_1, \dots, \omega_n)}{\partial(z_1, \dots, z_n)}$$

does not vanish at  $(a)$ . The inverse of (2),

$$(3) \quad z_i = \Omega_i(w_1, \dots, w_n) \quad (i = 1, \dots, n),$$

is then also regular at the point  $(w) = (b)$ , where

$$b_i = \omega_i(a_1, \dots, a_n) \quad (i = 1, \dots, n).$$

If the transformations (2) and (3) are analytic at the points  $(a)$  and  $(b)$  respectively, and if one is the inverse of the other, then each is regular. For the product of their jacobians is unity, and hence neither jacobian can vanish. By virtue of a theorem due to Clements,\* if the inverse of (2), the functions  $\omega_i$  being still analytic at  $(a)$ , is merely single-valued for points  $(w)$  which correspond to points  $(z)$  in the neighborhood of the point  $(z) = (a)$ , then (2) is regular.

It is evident that, if the functions  $F_i(z'_1, \dots, z'_n)$  satisfy *Condition A* at a point  $A' = (a')$ , and if we make a regular transformation

$$w'_i = \omega'_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n),$$

whereby the point  $(z') = (a')$  is carried into the point  $(w') = (b') = B'$ , the functions  $F_i$  going into  $\Phi_i$ :

$$F_i(z'_1, \dots, z'_n) = \Phi_i(w'_1, \dots, w'_n) \quad (i = 1, \dots, n),$$

then the functions  $\Phi_i$  will satisfy *Condition A* at the point  $B'$ , and the transformation

$$z_i = \Phi_i(w'_1, \dots, w'_n) \quad (i = 1, \dots, n),$$

will carry a region  $\tau'$  consisting of those points of a certain neighborhood

\* These Transactions, vol. 14 (1913), p. 328.

of  $B'$ , in which the functions  $\Phi_i(w'_1, \dots, w'_n)$  are all analytic, into a region  $\tau$  which lies in  $\sigma$ . The relation between the  $F_i$  and the  $\Phi_i$  is, of course, in this respect reciprocal.

The converse is also true, as we will presently show. First, however, a definition.

*Definition.* Let  $(F)$  and  $(\Phi)$  be two sets of functions which satisfy Condition A in the points  $A' = (a')$  and  $B' = (b')$  respectively. Then these two sets are said to be *equivalent* to each other in the points  $A'$  and  $B'$  if, the region  $\sigma'$  having been chosen arbitrarily small, it is then possible to find a region  $\tau'$  such that  $\tau$  lies in  $\sigma$ ; and reciprocally, the region  $\tau'$  having been taken arbitrarily small, there is then a region  $\sigma'$  such that  $\sigma$  lies in  $\tau$ .

On the other hand, the two sets of functions  $(F)$  and  $(\Phi)$  shall be said to be *distinct* if  $\sigma'$  and  $\tau'$  can be so chosen that  $\sigma$  and  $\tau$  have no point in common. This classification is not intended to be exhaustive.

**THEOREM.** *If  $F_i(z'_1, \dots, z'_n)$  and  $\Phi_i(w'_1, \dots, w'_n)$  are two sets of equivalent functions, and if we set*

$$(4) \quad F_i(z'_1, \dots, z'_n) = \Phi_i(w'_1, \dots, w'_n) \quad (i = 1, \dots, n),$$

*then these  $n$  equations define a regular transformation of the neighborhood of  $A'$  on the neighborhood of  $B'$ .*

In fact, there corresponds to each point  $(z')$  of a certain neighborhood  $\sigma'$  of  $A'$  through the equations

$$z_i = F_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n),$$

a point  $(z)$  of  $\sigma$ , and this point lies in  $\tau$ . As a point of the latter region it leads through the transformation

$$z_i = \Phi_i(w'_1, \dots, w'_n) \quad (i = 1, \dots, n),$$

to a point  $(w')$  of  $\tau'$ . Since the jacobian is each time different from 0, we thus obtain  $n$  functions

$$(5) \quad w'_i = \omega_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n),$$

each analytic in each point of  $\sigma'$ , and also finite in  $\sigma'$ . These functions can have, therefore, only removable singularities in the neighborhood of  $A'$ , and consequently they yield functions each analytic at  $A'$ .

Since the relation between the functions  $F_i$  and  $\Phi_i$  is reciprocal, we have thus proven that the equations (4) admit a solution with regard to the  $z'_i$ 's,

$$z'_i = \Omega_i(w'_1, \dots, w'_n) \quad (i = 1, \dots, n),$$

such that  $\Omega_i$  is analytic at  $B'$ , and hence the transformation (5) is regular at  $A'$ .

**COROLLARY.** If  $A'_1$  and  $B'_1$  are any two points of suitably chosen neighborhoods of  $A'$  and  $B'$  respectively, lying on the boundary of  $\sigma'$  and  $\tau'$ , then the above functions  $(F)$  and  $(\Phi)$  are either equivalent or distinct in  $A'_1$  and  $B'_1$ .

The functions will be equivalent if  $A'_1$  and  $B'_1$  correspond to each other by virtue of the transformation (5); otherwise the functions will be distinct.

**Condition B.** Two sets of functions  $(F)$  and  $(\Phi)$  which satisfy Condition A shall be said to satisfy Condition B if they are either equivalent or distinct.

More precisely, they satisfy Condition B in the points  $A'$  and  $B'$ . It follows, then, at once that if  $A'_1$  and  $B'_1$  be arbitrary points of certain neighborhoods of  $A'$  and  $B'$  respectively, lying on the boundary of  $\sigma'$  and  $\tau'$ , then the functions  $(F)$  and  $(\Phi)$  also satisfy Condition B in the points  $A'_1$  and  $B'_1$ .

It is clear that when the above functions  $(F)$  are considered first at  $A'$  and then at  $A'_1$ , they necessarily yield two sets of functions satisfying Condition B at these two points; for they are distinct.

**Condition C.** Let  $m$  sets of functions  $(F^{(k)})$ ,  $k = 1, \dots, m$ , be given which satisfy Condition A respectively in the points  $A'^{(k)}$ , and let  $\sigma'^{(k)}$  denote definite regions  $\sigma'$  corresponding respectively to each of these points. Moreover, let it be possible each time to find a larger region  $\bar{\sigma}'^{(k)}$  such that the corresponding region  $\bar{\sigma}'^{(k)}$  contains  $\sigma'^{(k)}$  wholly in its interior.

Let  $A_1'^{(p)}$  and  $A_1'^{(q)}$  be any two boundary points of  $\bar{\sigma}'^{(p)}$  and  $\bar{\sigma}'^{(q)}$  which are interior to the corresponding regions  $\bar{\sigma}'^{(p)}$  and  $\bar{\sigma}'^{(q)}$ . Then  $(F^{(p)})$  and  $(F^{(q)})$  shall satisfy Condition B in the points  $A_1'^{(p)}$ ,  $A_1'^{(q)}$ .

The above functions  $(F^{(k)})$  shall be said to satisfy Condition C if every (finite) point outside a certain hypersphere, i. e., every point  $(z_1, \dots, z_n)$ , for which

$$G < |z_1|^2 + \dots + |z_n|^2,$$

lies in some region  $\sigma^{(k)}$ .

**Functions of the Class  $\mathfrak{R}$ .** We are now in a position to define the functions of the class which we wish to use. Let  $(F^{(k)})$ ,  $k = 1, \dots, m$ , be  $m$  sets of functions which satisfy Condition C, each set being considered only in the points of its region  $\bar{\sigma}'^{(k)}$ . Let  $(\Phi)$  be a set of functions satisfying Condition A in a point  $B'$ , and let  $(\Phi)$  be equivalent with some  $(F^{(k)})$  in a boundary-point  $A_1'^{(k)}$  of  $\sigma'^{(k)}$ . The totality of such functions  $(\Phi)$ , each considered in a certain neighborhood of its point  $B'$ , constitutes the *Class of Functions  $\mathfrak{R}$* .

It is obvious that the class includes each set of functions  $(F^{(k)})$  considered not merely with reference to  $A'^{(k)}$ , but also with reference to any other point  $A_1'^{(k)}$  on the boundary of  $\sigma'^{(k)}$ , in which the functions  $F_i^{(k)}$  are not all analytic.

## 2. DEFINITION OF AN INFINITE REGION

By *ordinary* complex space we mean the finite space  $R$  of algebraic geometry. Its points are given by  $n$  complex coordinates,  $(z_1, \dots, z_n)$ . It is some-

times desirable to extend this space by the adjunction of ideal points,—the “points at infinity.” These points shall be introduced as follows.

By an *infinite region* shall be understood a class of elements  $U = \{P\}$  related among themselves and to the points of ordinary space by the following definitions. The elements  $P$  shall be referred to as *points at infinity*, or briefly *points*. As a matter of notation we write  $P = (A', F)$ ,  $Q = (B', \Phi)$ , etc.

*Definition i).* A point  $P$  of  $U$  is given by a finite point

$$A' = (a'_1, \dots, a'_n) = (a')$$

and a set of functions  $(F)$  which belong to an assigned class  $\mathfrak{R}$  and satisfy Conditions  $A$  in the point  $A'$ .

*Definition ii).* Two points of  $U$ ,

$$P = (A', F), \quad Q = (B', \Phi),$$

*coincide* (or are *identical*, the *same point*) if and only if the two sets of functions  $(F)$  and  $(\Phi)$  are equivalent in the points  $A'$  and  $B'$ .

*Definition iii).* The infinite point  $P = (A', F)$  *corresponds* to the finite point  $A'$  by virtue of the transformation

$$(6) \quad z_i = F_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n);$$

and conversely,  $A'$  corresponds to  $P$ . Similarly, an infinite point  $P$  corresponds to a second infinite point  $Q$  if both correspond to the same finite point  $A'$ . Any two points at infinity can be made to correspond to each other.

*Definition iv).* By the *neighborhood* of an infinite point  $P = (A', F)$  is meant the region  $\sigma$  and such points of  $U$  as correspond by (6) to boundary points of  $\sigma'$  which lie in  $s'$ .

A point  $P$  of  $U$  is said to be a *cluster-point* of a set of finite or infinite points if in every neighborhood of  $P$  there are points of the set distinct from  $P$ .

Two manifolds  $\mathfrak{M}_1$  and  $\mathfrak{M}_2$  *meet* or *intersect* in a point  $P$  of  $U$  if  $P$  is a cluster-point for points of each set.

This completes the definition of an infinite region, and we turn now to a theorem of fundamental importance.

**THEOREM 1.** *An arbitrary straight line of ordinary space,*

$$(7) \quad z_i = c_i + \lambda_i t \quad (i = 1, \dots, n),$$

$$0 < \sum_{i=1}^n |\lambda_i|,$$

*where  $t$  ranges over the whole finite  $t$ -plane, meets an infinite region  $U$  in one point  $P$ , and in no second point  $Q$  distinct from  $P$ .*

There is obviously at least one region  $\sigma^{(k)}$  which contains points of the

line (7) for which, no matter how large the positive number  $M$  be chosen,  $|t| > M$ . The images ( $z'$ ) of these points ( $z$ ) by the transformation

$$(8) \quad z_i = F_i^{(k)}(z'_1, \dots, z'_n) \quad (i = 1, \dots, n),$$

lie in  $\sigma'^{(k)}$  and have a cluster-point  $A'_k = (a'_1, \dots, a'_n)$  interior to  $\bar{s}'^{(k)}$  (though possibly on the boundary of  $s'^{(k)}$ ). To this point  $A'_k$  corresponds by  $(F^{(k)})$  a point  $P = (A'_k, F^{(k)})$  of  $U$  in which the line (7) meets  $U$ , and thus the first part of the proposition is established.

Let

$$t' = \frac{1}{t}, \quad F_i^{(k)} = \frac{H_i(z'_1, \dots, z'_n)}{G_i(z'_1, \dots, z'_n)}, \quad (i = 1, \dots, n).$$

On substituting for  $z_i$  in (8) its value from (7) we have

$$c_i + \frac{\lambda_i}{t'} = \frac{H_i}{G_i}$$

or

$$(9) \quad \lambda_i G_i(z'_1, \dots, z'_n) - t' [H_i(z'_1, \dots, z'_n) - c_i G_i(z'_1, \dots, z'_n)] = 0.$$

These equations are satisfied, in particular, by an infinite set of values

$$(t'^{(\mu)}, z_1'^{(\mu)}, \dots, z_n'^{(\mu)}), \quad (t'^{(\mu)} \neq 0),$$

where

$$\lim_{\mu \rightarrow \infty} t'^{(\mu)} = 0$$

and the points  $(z_1'^{(\mu)}, \dots, z_n'^{(\mu)})$  have  $A'_k$  as a cluster-point.

We have, then, in (9) a system of  $n$  equations in the  $n + 1$  variables  $t'$ ,  $z'_1, \dots, z'_n$ , to which Weierstrass's second implicit function theorem\* applies. From that theorem we infer that the simultaneous solutions of (9) in the neighborhood of the point  $(t', z'_1, \dots, z'_n) = (0, a'_1, \dots, a'_n) = (0, a')$  are given by one or more algebroid equations of the form

$$(10) \quad v_{m+1}^\nu + C_1 v_{m+1}^{\nu-1} + \dots + C_\nu = 0,$$

where  $C_q(v_1, \dots, v_m)$ ,  $q = 1, \dots, \nu$ , is analytic in the point  $(v) = (0)$  and vanishes there, and where  $v_{m+2}, \dots, v_{n+1}$  are single-valued functions on the configuration (10); the variables  $v_1, \dots, v_{n+1}$  being linear functions of  $t', z'_1, \dots, z'_n$ .

But  $m$  must equal 1, since otherwise to a given  $t'$  near 0 there would correspond more than one point ( $z'$ ) in  $\sigma'$  and hence more than one point ( $z$ ) in  $\sigma$ .

For the same reason,  $\nu$  must equal 1, too, and the number of algebroid equations (10) must also reduce to one. Hence  $t'$  itself can be taken as the inde-

\* Weierstrass, *Werke*, 3, pp. 79-80; *Madison Colloquium*, p. 192.

pendent variable, and the complete system of simultaneous solutions of (9) is given by  $n$  equations

$$z'_i = f_i(t') \quad (i = 1, \dots, n),$$

where  $f_i(t')$  is analytic in the point  $t' = 0$  and  $f_i(0) = a'_i$ .

From this last result we see that, when an arbitrarily restricted neighborhood  $s$  of the point  $P = (A'_k, F^{(k)})$  has been chosen in advance, a value of  $M$  can then be assigned such that all points  $(z)$  of (7), for which  $|t| > M$ , lie in  $s$  (and of course also in  $\sigma_k$ ).

The foregoing analysis has not only established the above theorem, but it yields the further result:

**THEOREM 2.** *If  $P = (A', F)$  be the point in which the line (7) meets the infinite region  $U$ , and if the neighborhood  $s$  of  $P$  be transformed by (1) on the neighborhood  $s'$  of the finite point  $A' = (a'_1, \dots, a'_n)$ , then the line (7) goes over into a curve which is analytic at  $(a')$  and in the neighborhood of this point lies, with the exception of the single point  $(a')$  itself, in the region  $\sigma'$ .*

### 3. THE EXTENSION OF WEIERSTRASS'S THEOREM

**Definition.** A function  $\Omega(z_1, \dots, z_n)$  shall be said to be *analytic* or *meromorphic* in a point  $P = (A', F)$  of the infinite region  $U$  if on making the transformation (1)  $\Omega$  is carried over into a function  $\Omega'(z'_1, \dots, z'_n)$  which is analytic or meromorphic at the point  $A'$ .

**THEOREM.** *Let ordinary space be closed by an infinite region  $U$  given by the above definitions, and let  $\Omega(z_1, \dots, z_n)$  be a function which is meromorphic at every point of the extended space. Then  $\Omega$  is a rational function.*

The proof can be given by means of the following theorem,\* which follows at once from the reasoning Hurwitz employed to prove Weierstrass's theorem in the space of analysis: If  $f(z_1, \dots, z_n)$  is a rational function of each individual variable, when all the others are assigned arbitrary values in the neighborhood of a certain fixed point,  $O$ , and if  $f$  is analytic in all the variables at  $O$ , then  $f$  is rational in all its arguments.

Let  $\Omega$  be analytic at the origin—more specifically, throughout the region

$$|z_i| < h \quad (i = 1, \dots, n).$$

Let  $(c)$  be a point of this region, and consider the function of  $z_k$  alone,

$$(11) \quad \Omega(c_1, \dots, c_{k-1}, z_k, c_{k+1}, \dots, c_n).$$

This function is analytic at the point  $z_k = 0$ . Let  $z_k = \alpha$  be the nearest singular point. Then it is readily shown that the function (11) has a pole at  $\alpha$ . For, in the neighborhood of the point  $(c_1, \dots, c_{k-1}, \alpha, c_{k+1}, \dots, c_n)$ ,

\* Cf. *Madison Colloquium*, p. 145.



we have

$$\Omega(z_1, \dots, z_n) = \frac{h_0 + h_1(z_k - \alpha) + \dots}{g_0 + g_1(z_k - \alpha) + \dots},$$

where  $g_i$  and  $h_i$  are functions of  $(z_1, \dots, z_{k-1}, z_{k+1}, \dots, z_n)$ , each analytic at the point  $(c_1, \dots, c_{k-1}, c_{k+1}, \dots, c_n)$ , and where  $g_0$  vanishes at this point. There must, however, be a  $g_m$  that does not vanish there, since otherwise, the fraction being assumed to be in its lowest terms, every point  $\alpha'$  of the neighborhood of  $\alpha$  would lead to a singular point  $(c_1, \dots, c_{k-1}, \alpha', c_{k+1}, \dots, c_n)$  of the function  $\Omega(z_1, \dots, z_n)$ , and hence it would follow that for *every* value of  $\alpha'$  the above point would be singular. But when  $\alpha' = 0$ ,  $\Omega(z_1, \dots, z_n)$  is analytic in the point in question.

Hence it appears that the function (11) has no other singular points except poles in the finite  $z_k$ -plane.

Finally, consider the point  $P = (A', F)$ , in which the line

$$(12) \quad z_i = c_i \quad (i = 1, \dots, k-1, k+1, \dots, n),$$

meets the infinite region  $U$ . By hypothesis,  $\Omega$  goes over by the transformation (1) into a function  $\Omega'(z'_1, \dots, z'_n)$  meromorphic at  $A'$ , and by the last theorem of § 2 the line goes over by the same transformation into a curve analytic at  $A'$  and lying, except for  $A'$ , wholly in  $\sigma'$ . Moreover, the function  $\Omega'(z'_1, \dots, z'_n)$  has no second singular point of the neighborhood of  $A'$  in common with this curve, since otherwise every point of the curve in this neighborhood would be a singular point of  $\Omega'$ , and hence every point of (12) would be a singular point of  $\Omega(z_1, \dots, z_n)$ . It follows, then, that the function  $\Omega'$ , taken along the curve in question, either remains finite or becomes infinite at  $A'$ , and hence the function (11) is either analytic at the point  $z_k = \infty$  or else has a pole there.

Hence the function (11) is rational in  $z_k$ , and all the hypotheses of the foregoing lemma are fulfilled. The function  $\Omega(z_1, \dots, z_n)$  is, therefore, rational in all  $n$  arguments.

#### 4. ON THE TRANSFORMATIONS OF A CLOSED SPACE INTO ITSELF

*Regular Transformations.* We have already laid down a definition according to which a transformation (2) is said to be regular at a finite point  $A$ ,  $(z) = (a)$ , the transformed point  $B$ ,  $(w) = (b)$ , also being finite. The definition shall now be extended to the case that one or both of the points  $A$ ,  $B$  are at infinity.

Let  $P = (A', F)$  be a point of  $U$ ; let  $B$ ,  $(w) = (b)$ , be a finite point; and let  $T$  be a transformation whereby the points of a certain neighborhood of  $P$  are referred in a one-to-one manner to the points of a corresponding neigh-

borhood of  $B$ . Then, through the transformation (1), a one-to-one correspondence  $\mathfrak{T}$  is established between the points of the above neighborhood of  $B$  and the points of a certain neighborhood of  $A'$ . And now the transformation  $T$  shall be said to be *regular* at  $P$  (and its inverse *regular* at  $B$ ) if  $\mathfrak{T}$  is regular, i. e., if  $\mathfrak{T}$  can be expressed by equations of the form

$$w_i = \phi_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n),$$

where the latter transformation is regular at  $A'$  and carries this point into  $B$ .

Finally, let  $P = (A', F)$  and  $Q = (B', \Phi)$  both be points of  $U$ , and let  $T$  be a transformation whereby the points of a certain neighborhood  $s$  of  $P$  are referred in a one-to-one manner to the points of a certain corresponding neighborhood  $t$  of  $Q$ . Then a transformation  $\mathfrak{T}$  of a certain neighborhood  $s'$  of  $A'$  on a certain neighborhood  $t'$  of  $B'$  is herewith defined. And now  $T$  shall be said to be *regular* at  $P$  if  $\mathfrak{T}$  is regular at  $A'$ , and  $B'$  corresponds to  $A'$ . The inverse transformation will then be regular at  $Q$ .

It follows at once from the foregoing definitions that if a transformation  $T$  is regular at a finite or infinite point  $P$  of the  $(z)$  space, and is given in the neighborhood of  $P$  by the equations

$$w_i = \omega_i(z_1, \dots, z_n),$$

then the functions  $\omega_i$  are meromorphic at  $P$ . If, furthermore, the inverse transformation be represented by

$$z_i = \Omega_i(w_1, \dots, w_n),$$

and if  $P = (A', F)$  be a point at infinity, while its image  $B$  is finite, then the functions  $\Omega_i$  satisfy Condition  $B$  and

$$P = (B, \Omega).$$

Lastly, if  $P = (A', F)$  and  $Q = (B', \Phi)$  are both at infinity, then  $Q$  can be represented in the form  $Q = (A', G)$ , where the points  $(z)$  and  $(w)$  given by the equations

$$z_i = F_i(z'_1, \dots, z'_n), \quad w_i = G_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n)$$

are the images of each other under  $T$ .

**THEOREM 1.** *Let ordinary space be closed by an infinite region according to the foregoing definitions. Let  $T$  be a transformation of the extended space into itself which is regular at every point. Then  $T$  is one-to-one without exception, and is expressed by equations of the form*

$$(I) \quad w_i = \omega_i(z_1, \dots, z_n) \quad (i = 1, \dots, n),$$

where  $\omega_i$  is rational in all  $n$  arguments, the inverse transformation

$$(II) \quad z_i = \Omega_i(w_1, \dots, w_n), \quad (i = 1, \dots, n),$$

also being rational.

Since the extended space is linearly simply connected and closed, it follows that the transformation  $T$  must be one-to-one without exception.

It is clear that there exists a pair of finite points  $A'$  and  $B'$ , images the one of the other, at which  $T$  is regular:

$$(13) \quad w_i = \omega_i(z_1, \dots, z_n) \quad (i = 1, \dots, n).$$

Let  $(Z')$  be an arbitrary finite point, and let  $L$  be a regular curve of finite space connecting  $A'$  with  $(Z')$ . Then each  $\omega_i(z_1, \dots, z_n)$  can be continued meromorphically along  $L$  to  $(Z')$ . If this were not the case, let  $(c')$  be the first point of  $L$  encountered in going from  $A'$  to  $(Z')$ , at which some  $\omega_i$  cannot be continued further meromorphically. But  $T$  is expressed in the neighborhood of  $(c')$  by equations of the type (13), the right-hand sides being meromorphic at  $(c')$ . Hence we are led to a contradiction.

Thus it appears that each function  $\omega_i$  in (13) can be continued meromorphically over the entire finite space. Finally, since the transformation  $T$  is to be regular in each point of the infinite region  $U$ , each function  $\omega_i$  must be meromorphic in such a point also, and the theorem is established.

COROLLARY. In no finite point  $(z) = (a)$ , in which all the functions  $\omega_i$  are analytic, can the jacobian

$$\frac{\partial(\omega_1, \dots, \omega_n)}{\partial(z_1, \dots, z_n)}$$

vanish.

Moreover, if  $P = (A', F)$  be an infinite point and if  $\omega'_i(z'_1, \dots, z'_n)$  denotes the function into which  $\omega_i(z_1, \dots, z_n)$  is transformed by (1); if finally  $P$  is carried by  $T$  into a finite point; then  $\omega'_i(z'_1, \dots, z'_n)$  is analytic at  $A'$ , and

$$\frac{\partial(\omega'_1, \dots, \omega'_n)}{\partial(z'_1, \dots, z'_n)}$$

does not vanish at  $(a')$ .

Lastly, if  $P$  is carried by  $T$  into a point  $Q = (B', \Phi)$  of  $U$ , and if

$$w'_i = \bar{\omega}_i(z'_1, \dots, z'_n) \quad (i = 1, \dots, n)$$

denotes the relation between the points of the neighborhoods of  $A'$  and  $B'$  corresponding to  $T$ , then

$$\frac{\partial(\bar{\omega}_1, \dots, \bar{\omega}_n)}{\partial(z'_1, \dots, z'_n)}$$

does not vanish at  $A'$ .

We can proceed now to a converse proposition, which we will state as follows.

THEOREM 2. *Let ordinary space be extended by an infinite region  $U$  according to the foregoing definitions, and let  $T$  be a transformation of the extended space into itself such that the jacobian relations of the preceding corollary are fulfilled at every point. Then  $T$  is regular at every point.*

For, such a closed space is linearly simply connected, and the above jacobian conditions are sufficient that *im Kleinen* the transformation be everywhere regular.

It appears, then, that if ordinary space be extended in a second way by an infinite region  $U'$  according to the foregoing definitions, and if  $T'$ :

$$w_i = \varpi_i(z_1, \dots, z_n) \quad (i = 1, \dots, n),$$

be a transformation of the second space into itself regular at every point, then  $T'$ , applied to the first space, will yield a transformation one-to-one in general, but with fundamental manifolds.

The question presents itself: Can an arbitrary birational transformation of ordinary space become, on extending space suitably by an infinite region, a transformation of the extended space into itself, which is regular at every point?

A first necessary condition that this be the case is that the jacobian shall not vanish at any finite point at which all the functions  $\omega_i(z_1, \dots, z_n)$  are analytic; and a similar condition must hold for the inverse transformation. It remains to examine the further jacobian conditions corresponding to the foregoing corollary. These conditions bear both on the given transformation and on the particular infinite region introduced.

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